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On the fine spectrum of the operator $B(r, s, t)$ over c_0 and c

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Abstract

We determine the fine spectrum of the operator $B(r, s, t)$ defined by a triple-band matrix over the sequence spaces c_0 and c. This generalizes the spectrum of the second-order difference operator Δ^2 and includes some other special cases such as the generalized difference operator $B(r, s)$ of [B. Altay, F. Başar, On the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces c_0 and c, Internat. J. Math. Math. Sci. 18 (2005) 3005–3013], the difference operator Δ of [B. Altay, F. Başar, On the fine spectrum of the difference operator on c_0 and c, Inform. Sci. 168 (2004) 217–224], the right shift and Zweier matrices.

Keywords: Spectrum of an operator; Difference operator; Generalized difference operator; Spectral mapping theorem; The sequence spaces c_0 and c

1. Preliminaries, background and notation

Let X and Y be the Banach spaces and $T : X \to Y$ be a bounded linear operator. By $R(T)$, we denote the range of T, i.e.,

$$R(T) = \{y \in Y : y = Tx, x \in X\}.$$

By $B(X)$, we also denote the set of all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$ then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^* f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$.

Let $X \neq \{\theta\}$ be a non-trivial complex normed space and $T : \mathcal{D}(T) \to X$ a linear operator defined on subspace $\mathcal{D}(T) \subseteq X$. We do not assume that $\mathcal{D}(T)$ is dense in X, or that T has closed graph $\{(x, Tx) : x \in \mathcal{D}(T)\} \subseteq X \times X$. We mean by the expression "T is *invertible*" that there exists a bounded linear operator $S : R(T) \to X$ for which $ST = I$ on $D(T)$ and $\overline{R(T)} = X$; such that $S = T^{-1}$ is necessarily uniquely determined, and linear; the boundedness of S means that T must be *bounded below*, in the sense that there is $k > 0$ for which $\|Tx\| \geq k\|x\|$ for all $x \in D(T)$. Associated with each complex number α is a perturbed operator

$$T_\alpha = T - \alpha I,$$

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defined on the same domain $\mathcal{D}(T)$ as T. The *spectrum* $\sigma(T, X)$ consists of those $\alpha \in \mathbb{C}$ for which T_α is not invertible, and the *resolvent* is the mapping from the complement $\sigma(T, X)$ of the spectrum into the algebra of bounded linear operators on X defined by $\alpha \longmapsto T_\alpha^{-1}$.

The name *resolvent* is appropriate, since T_α^{-1} helps to solve the equation $T_\alpha x = y$. Thus, $x = T_\alpha^{-1} y$ provided T_α^{-1} exists. More important, the investigation of properties of T_α^{-1} will be basic for an understanding of the operator T itself. Naturally, many properties of T_α and T_α^{-1} depend on α, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all α's in the complex plane such that T_α^{-1} exists. Boundedness of T_α^{-1} is another property that will be essential. We shall also ask for what α's the domain of T_α^{-1} is dense in X, to name just a few aspects. A *regular value* α of T is a complex number such that T_α^{-1} exists and is bounded and whose domain is dense in X. For our investigation of T, T_α and T_α^{-1}, we need some basic concepts in spectral theory which are given as follows (see [1, pp. 370–371]):

The *resolvent set* $\rho(T, X)$ of T is the set of all regular values α of T. Furthermore, the spectrum $\sigma(T, X)$ is partitioned into three disjoint sets as follows:

The *point (discrete) spectrum* $\sigma_p(T, X)$ is the set such that T_α^{-1} does not exist. An $\alpha \in \sigma_p(T, X)$ is called an *eigenvalue* of T.

The *continuous spectrum* $\sigma_c(T, X)$ is the set such that T_α^{-1} exists and is bounded and the domain of T_α^{-1} is dense in X.

The *residual spectrum* $\sigma_r(T, X)$ is the set such that T_α^{-1} exists (and may be bounded or not) but the domain of T_α^{-1} is not dense in X.

To avoid trivial misunderstandings, let us say that some of the sets defined above may be empty. This is an existence problem which we shall have to discuss. Indeed, it is well known that $\sigma_c(T, X) = \sigma_r(T, X) = \emptyset$ and the spectrum $\sigma(T, X)$ consists of only the set $\sigma_p(T, X)$ in the finite dimensional case.

From Goldberg [2, pp. 58–71], if X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$ and T^{-1}:

(I) $R(T) = X$,
(II) $R(T) \neq \overline{R(T)} = X$,
(III) $\overline{R(T)} \neq X$

and

(1) T^{-1} exists and is continuous,
(2) T^{-1} exists but is discontinuous,
(3) T^{-1} does not exist.

Applying Goldberg's classification to T_α, we have three possibilities for T_α and T_α^{-1};

(I) T_α is surjective,
(II) $R(T_\alpha) \neq \overline{R(T_\alpha)} = X$,
(III) $\overline{R(T_\alpha)} \neq X$

and

(1) T_α is injective and T_α^{-1} is continuous,
(2) T_α is injective and T_α^{-1} is discontinuous,
(3) T_α is not injective.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: $I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2$ and III_3. If α is a complex number such that $T_\alpha \in I_1$ or $T_\alpha \in II_1$ then α is in the resolvent set $\rho(T, X)$ of T. The further classification gives rise to the fine spectrum of T. If an operator is in state II_2 for example, then $R(T) \neq \overline{R(T)} = X$ and T^{-1} exists but is discontinuous and we write $\alpha \in II_2\sigma(T, X)$.

By a *sequence space*, we understand a linear subspace of the space $w = \mathbb{C}^{\mathbb{N}}$ of all complex sequences which contains ϕ, the set of all finitely non-zero sequences, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. We write ℓ_∞, c, c_0 and bv for the spaces of all bounded, convergent, null and bounded variation sequences, respectively. Also by ℓ_p, we denote the space of all p-absolutely summable sequences, where $1 \leq p < \infty$.

Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n, k \in \mathbb{N}$, and write

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad (n \in \mathbb{N}, x \in D_{00}(A)), \tag{1.1}$$

where $D_{00}(A)$ denotes the subspace of w consisting of $x \in w$ for which the sum exists as a finite sum. More generally if μ is a normed sequence space, we can write $D_\mu(A)$ for the $x \in w$ for which the sum in (1.1) converges in the norm of μ. We shall write

$$(\lambda : \mu) = \{A : \lambda \subseteq D_\mu(A)\}$$

for the space of those matrices which send the whole of the sequence space λ into μ in this sense. Our main focus in this paper is on the triple-band matrix $A = B(r, s, t)$, where

$$B(r,s,t) = \begin{bmatrix} r & 0 & 0 & 0 & \cdots \\ s & r & 0 & 0 & \cdots \\ t & s & r & 0 & \cdots \\ 0 & t & s & r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We assume here and hereafter that s and t are complex parameters which do not simultaneously vanish.

We begin by determining when a matrix A induces a bounded operator from c to c.

Lemma 1.1 (*cf.* [3, *Theorem 1.3.6, p. 6*]). *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c)$ from c to itself if and only if*

(1) *the rows of A are in ℓ_1 and their ℓ_1 norms are bounded,*
(2) *the columns of A are in c,*
(3) *the sequence of row sums of A is in c.*

The operator norm of T is the supremum of the ℓ_1 norms of the rows.

Corollary 1.2. *$B(r, s, t) : c \to c$ is a bounded linear operator with $\|B(r, s, t)\|_{(c:c)} = |r| + |s| + |t|$.*

Lemma 1.3 (*cf.* [3, *Example 8.4.5A, p. 129*]). *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c_0)$ from c_0 to itself if and only if*

(1) *the rows of A are in ℓ_1 and their ℓ_1 norms are bounded,*
(2) *the columns of A are in c_0.*

The operator norm of T is the supremum of the ℓ_1 norms of the rows.

Corollary 1.4. *$B(r, s, t) : c_0 \to c_0$ is a bounded linear operator with*

$$\|B(r, s, t)\|_{(c_0:c_0)} = \|B(r, s, t)\|_{(c:c)}.$$

We summarize the knowledge in the existing literature concerning with the spectrum of the linear operators defined by some particular limitation matrices over some sequence spaces. Wenger [4] examined the fine spectrum of the integer power of the Cesàro operator in c and Rhoades [5] generalized this result to the weighted mean methods. The fine spectrum of the Cesàro operator on the sequence space ℓ_p has been studied by Gonzàlez [6], where $1 < p < \infty$. The spectra of the Cesàro operator on the sequence spaces c_0 and bv have also been investigated by Reade [7] and Okutoyi [8], respectively. The fine spectrum of the Rhally operators on the sequence spaces c_0 and c has been examined by Yıldırım [9]. The fine spectrum of the Cesàro operator on the sequence space c_0 has been studied by Akhmedov and Başar [10]. Recently, de Malafosse [11] and Altay and Başar [12] have respectively studied the spectrum and the fine spectrum of the difference operator on the sequence spaces s_r and c_0, c; where s_r denotes the Banach space of all sequences $x = (x_k)$ normed by

$$\|x\|_{s_r} = \sup_{k \in \mathbb{N}} \frac{|x_k|}{r^k}, \quad (r > 0).$$

More recently, the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces c_0 and c has been studied by Altay and Başar [13]. Also, the fine spectrum of the generalized difference operator $B(r,s)$ over the sequence spaces ℓ_1 and bv has been studied by Furkan, Bilgiç and Kayaduman [14].

In this work, our purpose is to determine the fine spectrum of the operator $B(r,s,t)$ over the sequence spaces c_0 and c.

2. The fine spectrum of the generalized triple-band matrix operator $B(r,s,t)$ on the sequence spaces c_0 and c

In this section, the fine spectrum of the operator $B(r,s,t)$ on the sequence spaces c_0 and c has been examined.

Before giving the main theorem we should give the following remark. In this work, here and in what follows, if z is a complex number then by $\sqrt{z}$ we always mean the square root of z with non-negative real part. If $\mathrm{Re}(\sqrt{z}) = 0$ then $\sqrt{z}$ represents the square root of z with $\mathrm{Im}(\sqrt{z}) \geq 0$. The same results are obtained if $\sqrt{z}$ represents the other square root.

Theorem 2.1. *Let s be a complex number such that $\sqrt{s^2} = -s$ and define the set S by*

$$S = \left\{ \alpha \in \mathbb{C} : \left| \frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}} \right| \leq 1 \right\}.$$

Then, $\sigma(B(r,s,t), c_0) = S$.

Proof. First we prove that $(B(r,s,t)-\alpha I)^{-1}$ exists and is in $B(c_0)$ for $\alpha \notin S$ and next the operator $B(r,s,t)-\alpha I$ is not invertible for $\alpha \in S$.

Without loss of generality we may assume $\sqrt{s^2} = -s$. Let $\alpha \notin S$. It is easy to see that we must have $\alpha \neq r$ and so $B(r,s,t)-\alpha I$ is triangle, and hence has an inverse. We can calculate that

$$(B(r,s,t)-\alpha I)^{-1} = \begin{bmatrix} a_1 & 0 & 0 & 0 & \dots \\ a_2 & a_1 & 0 & 0 & \dots \\ a_3 & a_2 & a_1 & 0 & \dots \\ a_4 & a_3 & a_2 & a_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where

$$a_1 = \frac{1}{r-\alpha}$$
$$a_2 = -\frac{s}{(r-\alpha)^2}$$
$$a_3 = \frac{s^2-(r-\alpha)t}{(r-\alpha)^3}$$
$$\vdots$$

In fact this sequence is obtained recursively by letting $a_1 = \frac{1}{(r-\alpha)}$, $a_2 = \frac{-s}{(r-\alpha)^2}$ and

$$a_n = \frac{-(sa_{n-1}+ta_{n-2})}{(r-\alpha)}, \quad (n \geq 3).$$

It is easy to verify that, for $n \geq 1$,

$$a_n = \frac{1}{\sqrt{s^2-4t(r-\alpha)}} \left\{ \left[\frac{-s+\sqrt{s^2-4t(r-\alpha)}}{2(r-\alpha)} \right]^n - \left[\frac{-s-\sqrt{s^2-4t(r-\alpha)}}{2(r-\alpha)} \right]^n \right\}. \tag{2.1}$$

Now by letting

$$u_1 = \frac{-s+\sqrt{s^2-4t(r-\alpha)}}{2(r-\alpha)} \quad \text{and} \quad u_2 = \frac{-s-\sqrt{s^2-4t(r-\alpha)}}{2(r-\alpha)}$$

we have

$$a_n = \frac{1}{\sqrt{s^2 - 4t(r-\alpha)}}({u_1}^n - {u_2}^n).$$

It should be noted here that if one assumes $\sqrt{s^2} = s$ then we obtain the same sequence.

If $4t(r-\alpha) = s^2$ then

$$a_n = \left(\frac{2n}{-s}\right)\left[\frac{-s}{2(r-\alpha)}\right]^n$$

and simple calculations give that $(a_n) \in \ell_1$ if and only if $\left|\frac{-s}{2(r-\alpha)}\right| < 1$. Therefore $\alpha \notin S$ implies $a_n \to 0$.

So we may assume that $4t(r-\alpha) \neq s^2$. Since $\alpha \notin S$ we have $|u_1| < 1$. Now we show that $|u_2| < 1$. Since $|u_1| < 1$ we have

$$\left|1 + \sqrt{1 + 4t(r-\alpha)/s^2}\right| < \left|\frac{2(r-\alpha)}{-s}\right|.$$

Since $|1 - \sqrt{z}| \leq |1 + \sqrt{z}|$ for any $z \in \mathbb{C}$ we must have

$$\left|1 - \sqrt{1 + 4t(r-\alpha)/s^2}\right| < \left|\frac{2(r-\alpha)}{-s}\right|$$

which leads us to the fact that $|u_2| < 1$.

This proves that $\alpha \notin S$ which implies the fact $a_n \to 0$ as $n \to \infty$. Now

$$\begin{aligned} \|(B(r,s,t) - \alpha I)^{-1}\|_{(c_o:c_o)} &= \sup_{n\in\mathbb{N}} \sum_{k=1}^{n} |a_k| = \sum_{k=1}^{\infty} |a_k| \\ &\leq \frac{1}{|\sqrt{s^2 - 4t(r-\alpha)}|}\left(\sum_{k=1}^{\infty} |u_1|^n + \sum_{k=1}^{\infty} |u_2|^n\right) \\ &< \infty \end{aligned} \tag{2.2}$$

since $|u_1| < 1$ and $|u_2| < 1$. This shows that $\sigma(B(r,s,t), c_0) \subseteq S$.

Suppose that $\alpha \in S$. If $\alpha = r$ then $B(r,s,t) - \alpha I$ is represented by the matrix

$$B(0,s,t) = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ s & 0 & 0 & 0 & \dots \\ t & s & 0 & 0 & \dots \\ 0 & t & s & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Since $B(r,s,t) - rI = B(0,s,t)$ does not have a dense range, it is not invertible.

If $s^2 = 4t(r-\alpha)$ then, for all $n \geq 1$, we get

$$a_n = \left(\frac{2n}{-s}\right)\left[\frac{-s}{2(r-\alpha)}\right]^n$$

in which case $(B(r,s,t) - \alpha I)^{-1}$ is not in $B(c_0)$ since $|-s/(2(r-\alpha))| \geq 1$.

So, we may assume that $\alpha \neq r$ and $s^2 \neq 4t(r-\alpha)$.

Now, since $\alpha \neq r$ then $B(r,s,t) - \alpha I$ is a triangle but since $s^2 \neq 4t(r-\alpha)$ we must have $|u_1| > |u_2|$, from which it follows that $a_n \nrightarrow 0$ and so $\sum_{k=1}^{\infty} |a_k|$ diverges.

Therefore $(B(r,s,t) - \alpha I)^{-1}$ is not in $B(c_0)$. This shows that S is a subset of $\sigma(B(r,s,t), c_0)$. This completes the proof. □

Theorem 2.2. $\sigma_p(B(r,s,t), c_0) = \emptyset$.

Proof. Suppose that $B(r,s,t)x = \alpha x$ for $x \neq \theta = (0,0,0,\ldots)$ in c_0. Then, by solving the system of linear equations

$$\left.\begin{array}{l} rx_0 = \alpha x_0 \\ sx_0 + rx_1 = \alpha x_1 \\ tx_0 + sx_1 + rx_2 = \alpha x_2 \\ tx_1 + sx_2 + rx_3 = \alpha x_3 \\ tx_2 + sx_3 + rx_4 = \alpha x_4 \\ \vdots \end{array}\right\}$$

we find that if x_{n_0} is the first non-zero entry of the sequence $x = (x_n)$ then from the equation

$$tx_{n_0-2} + sx_{n_0-1} + rx_{n_0} = \alpha x_{n_0}$$

we obtain that $\alpha = r$ and from the next equation we obtain $x_{n_0} = 0$. This contradicts the fact that $x_{n_0} \neq 0$ which completes the proof. $\square$

If $T : c_0 \to c_0$ is a bounded linear operator with the matrix A, then it is known that the adjoint operator $T^* : c_0^* \to c_0^*$ is defined by the transpose A^t of the matrix A. It should be noted that the dual space c_0^* of c_0 is isometrically isomorphic to the Banach space ℓ_1 of absolutely summable sequences normed by $\|x\| = \sum_k |x_k|$.

Theorem 2.3. *$\sigma_p(B(r,s,t)^*, c_0^*) = S_1$, where $S_1 = \{\alpha \in \mathbb{C} : |\frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}| < 1\}$.*

Proof. Suppose $B(r,s,t)^*x = \alpha x$ for $x \neq \theta$ in $c_0^* \cong \ell_1$. Consider the system of linear equations

$$\left.\begin{array}{l} rx_0 + sx_1 + tx_2 = \alpha x_0 \\ rx_1 + sx_2 + tx_3 = \alpha x_1 \\ rx_2 + sx_3 + tx_4 = \alpha x_2 \\ \vdots \end{array}\right\}.$$

It is clear that if $\alpha = r$ then we may choose $x_0 \neq 0$ and so $x = (x_0, 0, 0, \ldots)$ is an eigenvector corresponding to $\alpha = r$. Assume $\alpha \neq r$; then we obtain that

$$x_n = \frac{P_{n-1}}{t^{n-1}}(\alpha - r)x_0 + \frac{P_n}{t^{n-1}}x_1; \quad (n \geq 2), \tag{2.3}$$

where $P_n = a_n(r-\alpha)^n$. If α is a number such that

$$\left|\frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}\right| < 1$$

then we may choose $x_0 = 1$, $x_1 = \frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}$. Then one can verify that $x_2 = (x_1)^2, x_3 = (x_1)^3, \ldots, x_n = (x_1)^n, \ldots$ for all $n \geq 2$, and so $x \in \ell_1$ since $|x_1| < 1$. This shows that $S_1 \subseteq \sigma_p(B(r,s,t)^*, \ell_1)$.

Now assume α is a number such that

$$\left|\frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}}\right| \geq 1,$$

i.e. we have $|u_1| \leq 1$. We must show that $\alpha \notin \sigma_p(B(r,s,t)^*, \ell_1)$. Using (2.3) we obtain

$$\frac{x_{n+1}}{x_n} = \left(\frac{r-\alpha}{t}\right)\left(\frac{a_n}{a_{n-1}}\right)\left[\frac{-x_0 + \frac{a_{n+1}}{a_n}x_1}{-x_0 + \frac{a_n}{a_{n-1}}x_1}\right].$$

Notice that $\frac{r-\alpha}{t} = \frac{1}{u_1u_2}$. Now we examine three cases.

Case (i): $|u_2| < |u_1| \leq 1$.

In this case we have $s^2 \neq 4t(r-\alpha)$ and

$$\lim_n \frac{a_n}{a_{n-1}} = \lim_n \frac{a_{n+1}}{a_n} = \lim_n \frac{u_1^{n+1} - u_2^{n+1}}{u_1^n - u_2^n} = \lim_n \frac{u_1^{n+1}\left[1-(u_2/u_1)^{n+1}\right]}{u_1^n\left[1-(u_2/u_1)^n\right]} = u_1.$$

Now, if $-x_0 + u_1 x_1 = 0$ then we get $x_n = \frac{1}{u_1{}^n} x_0$ which is not in ℓ_1 since $|u_1| \leq 1$. Otherwise

$$\lim_n \left| \frac{x_{n+1}}{x_n} \right| = \frac{1}{|u_1||u_2|} |u_1| = \frac{1}{|u_2|} > 1.$$

Case (ii): $|u_2| = |u_1| < 1$.

In this case $s^2 = 4t(r - \alpha)$ and using the formula

$$a_n = \left(\frac{2n}{-s} \right) \left[\frac{-s}{2(r-\alpha)} \right]^n$$

we obtain

$$\lim_n \frac{a_{n+1}}{a_n} = \frac{-s}{2(r-\alpha)} = u_2 = u_1$$

and so

$$\lim_n \left| \frac{x_{n+1}}{x_n} \right| = \frac{1}{|u_1||u_2|} |u_1| = \frac{1}{|u_2|} > 1.$$

Case (iii): $|u_2| = |u_1| = 1$.

In this case $s^2 = 4t(r-\alpha)$ and so we have $\left| \frac{-s}{2t} \right| = 1$. Assume $\alpha \in \sigma_p(B(r, s, t)^*, \ell_1)$. This implies that $\theta \neq x \in \ell_1$. From (2.3) we have

$$x_n = \left(\frac{-s}{2t} \right)^{n-1} \left[(1-n) \left(\frac{-s}{2t} \right) x_0 + x_1 \right].$$

Since $\lim_n |x_n| = 0$ we must have $x_0 = x_1 = 0$. But this implies that $x = \theta$. This is a contradiction and so we must have $\alpha \notin \sigma_p(B(r, s, t)^*, \ell_1)$.

In case (i) and case (ii) above, by the d'Alembert test $x \notin \ell_1$. In case (iii) $\alpha \in \sigma_p(B(r, s, t)^*, \ell_1)$ leads to a contradiction. This is what we wished to prove. □

Now, we may give the following lemma required in the proof of theorems given in the present section, below:

Lemma 2.4 (*[2, p. 59]*). *T has a dense range if and only if T^* is one to one.*

Theorem 2.5. (i) $\sigma_r(B(r, s, t), c_0) = S_1$,
(ii) $\sigma_c(B(r, s, t), c_0) = S_2$,

where S_1 is defined as in Theorem 2.3 and $S_2 = \left\{ \alpha \in \mathbb{C} : \left| \frac{2(r-\alpha)}{-s+\sqrt{s^2-4t(r-\alpha)}} \right| = 1 \right\}$.

Proof. (i) Since $\sigma_p(B(r, s, t)^*, c_0^*) = S_1$ then $B(r, s, t)^* - \alpha I$ is not one to one for all $\alpha \in S_1$. Therefore, by Lemma 2.4, $B(r, s, t) - \alpha I$ does not have a dense range for all $\alpha \in S_1$.

(ii) Since $\sigma(B(r, s, t), c_0)$ is the disjoint union of the parts $\sigma_p(B(r, s, t), c_0)$, $\sigma_r(B(r, s, t), c_0)$ and $\sigma_c(B(r, s, t), c_0)$, we must have $\sigma_c(B(r, s, t), c_0) = S_2$. □

Theorem 2.6. *If $\alpha \in \sigma_c(B(r, s, t), c_0)$ then $\alpha \in II_2\sigma(B(r, s, t), c_0)$.*

Proof. By Theorem 2.1, if $s^2 = 4t(r-\alpha)$ then the inverse of $B(r, s, t) - \alpha I$ is discontinuous; and so has an unbounded inverse.

By Theorem 2.3, $B(r, s, t)^* - \alpha I$ is one to one. By Lemma 2.4 $B(r, s, t) - \alpha I$ has a dense range.

To verify that $B(r, s, t) - \alpha I$ is not surjective, it is sufficient to show that there is no sequence $x = (x_n) \in c_0$ such that $(B(r, s, t) - \alpha I)x = y$ for some $y \in c_0$. Let us consider the sequence $y = (1, 0, 0, \ldots) \in c_0$. This gives $x = (a_1, a_2, a_3, \ldots)$, where the a_i's are as in the proof of Theorem 2.1. Since α is the element of the spectrum, by Theorem 2.1, then $x \notin c_0$. This completes the proof. □

Theorem 2.7. *If $|t| < |s|$, then $r \in III_1\sigma(B(r, s, t), c_0)$. If $|t| \geq |s|$, then $r \in III_2\sigma(B(r, s, t), c_0)$.*

Proof. If $\alpha = r$ then, by Theorem 2.5(i), $B(r,s,t) - \alpha I$ is in state III_1 or III_2. A left inverse of $B(0,s,t)$ is

$$(B(0,s,t))^{-1} = \begin{bmatrix} 0 & \frac{1}{s} & 0 & 0 & \dots \\ 0 & \frac{-t}{s^2} & \frac{1}{s} & 0 & \dots \\ 0 & \frac{(-t)^2}{s^3} & \frac{(-t)}{s^2} & \frac{1}{s} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In other words $(B(0,s,t))^{-1} = (b_{nk})$ with

$$b_{nk} = \begin{cases} \dfrac{(-t)^{n+1-k}}{s^{n+2-k}}, & (1 < k < n+2) \\ 0 & (k = 1 \text{ or } k \geq n+2). \end{cases}$$

The matrix B is in $B(c_0)$ for $|t| < |s|$, and is not in $B(c_0)$ for $|t| \geq |s|$. That is $B(0,s,t)$ has a continuous inverse for $|t| < |s|$ but it does not have a continuous inverse for $|t| \geq |s|$. Therefore $r \in III_1\sigma(B(r,s,t), c_0)$ for $|t| < |s|$ and $r \in III_2\sigma(B(r,s,t), c_0)$ for $|t| \geq |s|$. This completes the proof. □

Theorem 2.8. *If $\alpha \neq r$ and $\alpha \in \sigma_r(B(r,s,t), c_0)$ then $\alpha \in III_2\sigma(B(r,s,t), c_0)$.*

Proof. By Theorem 2.5(i), $B(r,s,t) - \alpha I \in III_1$ or $\in III_2$. Hence by (2.2) the inverse of the operator $B(r,s,t) - \alpha I$ is discontinuous. Therefore $B(r,s,t) - \alpha I$ has an unbounded inverse. □

If $T : c \to c$ is a bounded matrix operator with matrix A, then $T^* : c^* \to c^*$ acting on $\mathbb{C} \oplus \ell_1$ has a matrix representation of the form

$$\begin{bmatrix} \chi & 0 \\ b & A^t \end{bmatrix};$$

where χ is the limit of the sequence of row sums of A minus the sum of the limit of the columns of A, b is the column vector whose kth entry is the limit of the kth column of A for each $k \in \mathbb{N}$, and the superscript t represents the transpose of a matrix. For $B(r,s,t) : c \to c$, the matrix $B(r,s,t)^* \in B(\ell_1)$ is of the form

$$B(r,s,t)^* = \begin{bmatrix} r+s+t & 0 \\ 0 & B(r,s,t)^t \end{bmatrix}.$$

Theorem 2.9. *$\sigma_p(B(r,s,t)^*, c^*) = S_1 \cup \{r+s+t\}$, where S_1 is defined as in Theorem 2.3.*

Proof. Suppose $B(r,s,t)^*x = \alpha x$ for $x \neq \theta$ in ℓ_1. Then by solving the system of linear equations

$$\left.\begin{array}{l} (r+s+t)x_0 = \alpha x_0 \\ rx_1 + sx_2 + tx_3 = \alpha x_1 \\ rx_2 + sx_3 + tx_4 = \alpha x_2 \\ rx_3 + sx_4 + tx_5 = \alpha x_3 \\ \vdots \end{array}\right\}$$

we obtain that

$$x_n = (r-\alpha)^n (a_n x_2 - a_{n-1} x_1); \quad (n \geq 3). \tag{2.4}$$

If $x_0 \neq 0$, then $\alpha = r+s+t$. So, $\alpha = r+s+t$ is an eigenvalue with the corresponding eigenvector $x = (x_0, 0, 0, \dots)$.

If $\alpha \neq r+s+t$, then $x_0 = 0$ and so using arguments similar to those in the proof of Theorem 2.3 one can see by (2.4) that $x \notin \ell_1$. This completes the proof. □

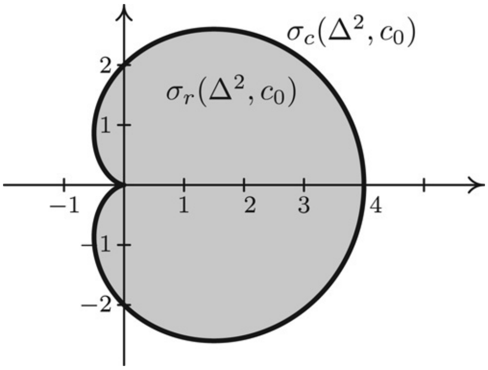

Fig. 1. The spectrum of Δ^2 over c_0 and c is inside the cardioid $r = 4\cos^2(\theta/2)$.

Since the fine spectrum of the operator $B(r, s, t)$ on c can be obtained using arguments similar to those used in the case of the space c_0, we omit the details and give the results without proof.

Theorem 2.10. (i) $\sigma(B(r, s, t), c) = S$,
(ii) $\sigma_p(B(r, s, t), c) = \emptyset$,
(iii) $\sigma_r(B(r, s, t), c) = \sigma_p(B(r, s, t)^*, c^*)$,
(iv) $\sigma_c(B(r, s, t), c) = S_2 \setminus \{r + s + t\}$,
(v) $\begin{cases} B(r,s,t) - \alpha I \in I_1, & \alpha \notin \sigma(B(r,s,t), c), \\ r \in III_1\sigma(B(r,s,t), c), & |t| < |s|, \\ r \in III_2\sigma(B(r,s,t), c), & |t| \geq |s|, \\ \alpha \in II_2\sigma(B(r,s,t), c), & \alpha \in \sigma_c(B(r,s,t), c), \\ \alpha \in III_2\sigma(B(r,s,t), c), & \alpha \in \sigma_r(B(r,s,t), c) \setminus \{r\} \end{cases}$

where S and S_2 are defined as in Theorems 2.1 and 2.5(ii), respectively.

Theorem 2.11. *$\sigma(B(r, s, t), \ell_\infty) = S$, where S is defined as in Theorem 2.1.*

Proof. It is known from Cartlidge [15] that if a matrix operator A is bounded on c then $\sigma(A, c) = \sigma(A, \ell_\infty)$. Now, the proof is immediate from Theorem 2.10(i) with $A = B(r, s, t)$. □

3. Conclusion

It is immediate that our results reduce to the spectrum of $B(r, s)$ studied by Altay and Başar [13], and to the spectrum of Δ determined by Altay and Başar [12] over the sequence spaces c_0 and c, since $B(r, s, 0) = B(r, s)$ and $B(1, -1, 0) = \Delta$, respectively. Additionally, the present work includes several special cases such as the right shift and Zweier matrices.

For this reason, our study is more general and more comprehensive than the previous works. Furthermore although by applying The Spectral Mapping Theorem for Polynomials (cf. [1, p. 381]) to Theorems 2.1 and 2.7 of Altay and Başar [12], the spectrum Δ^2 which is the composition of the operator Δ with itself over the sequence spaces c_0 an c may be derived as the set

$$\{\alpha \in \mathbb{C} : |1 - \sqrt{\alpha}| \leq 1\},$$

one can directly produce the same result from the present paper since $\Delta^2 = B(1, -2, 1)$. It is interesting that the spectrum of the operator Δ^2 over the sequence spaces c_0 and c is the region enclosed by the cardioid $r = 4\cos^2(\theta/2)$ and is represented by Fig. 1. Nevertheless, the spectrum of several special limitation matrices over the sequence spaces c_0 and c is the region enclosed by a circle (cf. [8,10,12,13]).

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